Geometry has traditionally been the course in which a deductive structure is first taught and formalized. This is tradition, but not good pedagogy. In a subject that is highly visual we ask students to deny what they see until they have performed this mysterious ritual called a two-column proof.

The van Hiele Model

Whenever there is discussion about proof in geometry we should first review the research of Dina van Hiele-Geldorf and her husband Pierre van Hiele. Their research resulted in the van Hiele model of geometric reasoning. Research by numerous mathematics educators including Usiskin 1982; Senk 1985; Burger and Shaughnessy 1986; Geddes and Fry 1988; Clements and Battista 1992; and more recently Battista 2007, support the accuracy of the van Hiele model. The model consists of five levels of geometric reasoning that students pass through from pre- deductive visual thinking to an understanding of formal proof and mathematical structures.

Level 0: ***Visualization*** –students can identify rectangles by sight but squares are squares and are not seen as special rectangles.

Level 1: ***Analysis*** (descriptive) – students can identify properties of rectangles (by drawing, measuring, and making models) but cannot yet derive other properties from given.

Level 2: ***Informal Deduction*** – students can give an informal argument to justify that the figure is a rectangle from given properties.

Level 3: ***Formal Deduction*** – students are capable of creating original logical arguments.

Level 4: ***Rigor*** – students are capable of reasoning about mathematical structures (i.e. Euclidean -vs.- non-Euclidean geometries).

The van Hiele model asserts these levels are sequential and hierarchical. That is, students cannot move to the next higher level until they have successful mastered the previous level. Furthermore movement from a lower level to the next higher level depends more on content and pedagogy than on maturity and age.

The van Hiele research shows students aren’t ready for formal proof until they have had concrete experiences with geometric figures and have successfully mastered earlier levels of visual thinking.

Research shows 70% of high school students enter geometry operating at level 0 or 1 on the van Hiele measure of geometric reasoning (Shaughnessy and Burger 1985; Senk 1989). Yet "traditional" geometry textbooks that begin with establishing postulates and proving theorems are expecting students to immediately begin their geometry experience at levels 3-4. When the teacher and the textbook are presenting geometry at van Hiele level 3 or higher, while the students are functioning at level 0 or 1, it should be no surprise that there is such a high failure rate in traditional geometry courses.

From *Mathematician's Delight* by English Mathematician W.W. Sawyer (1911-2008)

It would, I suppose, be quite possible to teach a deaf and dumb child to play the piano. When it played a wrong note, it would see the frown of its teacher, and try again. But it would obviously have no idea of what it was doing, or why anyone should devote hours to such an extraordinary exercise. It would have learnt an imitation of music. And it would fear the piano exactly as most students fear what is suppose to be mathematics.

Some geometry teachers claim they can successfully teach all of their geometry students how to create geometry proofs. Perhaps they only have those 30% that are entering geometry and functioning above van Hiele levels 0 and 1. I suspect however, they may also be teaching an "imitation geometry." Their students are trained to go through the motions of stating the theorem about to be proved (5 points), then stating the given (5 points), then stating the show (5 points), then drawing the diagram to the right (5 points), then drawing the big T (5 points), then putting the given information in the first few lines of the T-proof (5-15 points), then writing given to the right of each statement (5-15 points), then throwing in enough statements to garner enough points to get credit for the exercise without having any clue as to what he or she was doing or why anyone should devote hours to such an extraordinary exercise. They are doing imitation geometry.

Discovering Geometry, in its first edition, was an innovator in addressing students’ needs for gradual development of the deductive process. *Discovering Geometry* is the only high school geometry textbook on the market that is aligned with the van Hiele model and other research on geometric proof. We accept the vast majority of student are entering geometry at very low van Hiele levels of geometric reasoning and our goal, with careful deliberate scaffolding, is to move them to higher and higher levels of geometric reasoning as they progress through the course.

The Role and Function of Proof

Geometry student's consistent difficulties with understanding proofs (an international problem BTW) should not be solely attributed to their inability to reason but perhaps our inability to recognize there are many purposes for doing proofs and we have been stressing the wrong purposes for proof at inappropriate times. Professor Michael de Villiers' research on the role and function of proof identifies six basic roles for proof:

- **Verification –**to remove doubt, to convince someone of the truth of a statement
- **Systematization** –organize known results into a deductive system of postulates, definitions, and theorems
- **Explanation** –insight into why something is true
- **Discovery** –proof can occasionally lead to new unexpected results
- **Communication** –proof can create a forum for critical debate
- **Intellectual Challenge** –proof can be a testing ground for intellectual stamina and ingenuity

The function of proof in a high school geometry course has been mostly two-fold: to remove doubt, to convince someone of the truth of a statement –**verification** and to establish geometry as a mathematical system –**systematization**.

Many high school mathematics teachers seem to hold this naive view the main function of proof is to provide verification that a given statement is true. The role of proof as a means of verification is a useful method of verifying the "truth" of a proposition within a mathematical system, especially when coming across surprising (non-intuitive) results. However this view does not stand up to actual mathematician's experiences. Professor George Polya wrote, “When you have satisfied yourself that the theorem is true, you start proving it.” If the sole or primary purpose for doing proofs in a high school geometry course is to provide verification of the truth of a statement then students functioning below level 4 on the van Hiele scale will continue to question, or worse, disregard the process of proof.

If systematization is emphasized as a primary function of proof right from the start of a geometry course, the same poor results will persist. The van Hiele model tells us systematization requires the highest van Hiele level of geometric reasoning. Geometry textbooks that begin their geometry program with lists of definitions and postulates, and then begin doing two-column proofs are ignoring the research. Only an honors course or any class in which all students are finally operating at van Hiele level 4 would have any chance of success in a course that looks at geometry from the perspective of a mathematical system. Until some magic happens and all students beginning geometry enter the course functioning at van Hiele level 3, any attempt to create a mathematical system for a regular or informal geometry course is likely to continue to have major problems.

Students can acquire a very high degree of confidence in a conjecture arrived at by inductive methods but these methods may not explain why the conjecture is true. Here is where proof can come to the rescue. An example would be the inductive discovery that the sum of the measures of the three angles of a triangle is always 180°. A good inductive first approach would be to ask students to measure the three angles of a number of triangles thus gaining reasonable confidence the sum is indeed 180°. The same can be done with dynamic geometry software. Either investigative approach is a good first step because it is the first approach students would take. These inquiry approaches do give students confidence in their conjecture however they give no insight as to why the sum is always 180°. The investigation should be followed by a second investigation where they cut out the triangle and then tear off two of the angles and arrange them on both sides of the third angle to create a straight line. From this arrangement they can see the three angles create a line parallel to the third side. This visually explains what properties this conjecture is dependent upon and why the conjecture is true. This can also be done quickly with a patty paper investigation.

From *The Role and Function of Proof in Mathematics* by Michael de Villiers:

It is not a question of "*making sure,*" *but rather a question of* "*explaining why*."

Explanation as the Primary Role of Proof

Using proof as a means of explaining why something is true is the most meaningful role proof can play in a high school geometry course. Asking why something is true, after performing investigations that have convinced students it is true, is a powerful 1-2 punch. Explaining why can be an effective tool regardless of a student's van Hiele level. This is the approach we take with *Discovering Geometry*.

In practically every lesson in the fourth edition of *Discovering Geometry* (DG4) students are asked to perform geometric investigations and then make their geometry conjectures. After performing their investigation and making their conjecture they are asked, "can you explain why?" For example, after their very first two investigations leading to geometric conjectures, the Linear Pair Conjecture and the Vertical Angles Conjecture, students are asked:

"***Developing Proof*** You used inductive reasoning to discover both the Linear Pair Conjecture and the Vertical Angles Conjecture. Are they related? If you accept the Linear Pair Conjecture as true, can you use deductive reasoning to show that the Vertical Angles Conjecture must be true?"

We then ask them to work with their cooperative group members to develop a paragraph proof explaining why the conjecture is true then check their reasoning with ours.

Later after discovering the Triangle Sum Conjecture students are asked:

"***Developing Proof*** The investigation may have convinced you that the Triangle Sum Conjecture is true, but can you explain *why* it is true for every triangle?"

We then direct them to look back at the arrangement of the three angles they tore off and reassembled forming a line. We ask, "how is the resulting line related to the original triangle?" This is their lead-in to creating a paragraph proof explaining why their conjecture fits with what they have already discovered/proved about parallel lines.

The inductive and deductive reasoning in DG4 continues with investigating, conjecturing, and explaining why, from Chapter 2 through Chapter 12. It isn't until the last chapter, when there may be students ready for van Hiele level 4 reasoning, that we introduce geometry as a formal mathematical system.

Again from *Mathematician's Delight* by English Mathematician W.W. Sawyer (1911-2008)

The Great Pyramid was built in 3900 B.C. by rules based on practical experience: Euclid's system did not appear until 3,600 years later. *It is quite unfair to expect children to start studying geometry in the form that Euclid gave it. One cannot leap 3,600 years of human effort so lightly! The best way to learn geometry is to follow the road which the human race originally followed:* Do *things,* make *things,* notice *things,* arrange *things, and only then* –reason *about things.*

Once deductive arguments began to sprout up around Ancient Greece, it still took awhile for the process to be accepted. From the first deductive arguments by Thales to Euclid's Elements, it took over 300 years. We owe it to our students to give them time to move up through the van Hiele levels so they come to understand the role proof plays in mathematics.